King Fahd University of Petroleum & Minerals,
Dhahran 31261, Saudi Arabia

Information and Computer Science Department

ICS-252: Discrete Structures (Fall 2000)

Article 4.2: Solved Exercises (Pigeonhole Principle)

12. How many ordered pairs of integers (a, b) are needed to guarantee that there are two ordered pairs (a_1, b_1) and (a_2,b_2) such that a_1 mod 5 = a_2 mod 5 and b_1 mod 5 = b_2 mod 5?

Division by 5 gives 5 remainders 0, 1, 2, 3, 4.

Both a mod 5 and b mod 5 both can have 5 different values in the ordered pair (a, b). Hence we would have 5.5 = 25 unique pairs of remainders. (For example (0,0), (0,1), (0,2), (0,3), (0,4), (1,0), (1,1), …. , (4,4)).

By the pigeonhole principle we need 25 + 1 = 26 ordered pairs in order to guarantee that there are two ordered pairs (a_1, b_1) and (a_2,b_2) such that a_1 mod 5 = a_2 mod 5 and b_1 mod 5 = b_2 mod 5.

16. There are 51 houses on a street. Each house has an address between 1000 and 1099 inclusive. Show that at least two houses have addresses that are consecutive integers.

There are 100 addresses between 1000 and 1099 (from 0 to 99). Half of these must be even integers (1000, 1002, 1004, …) and half of them must be odd (1001, 1003, 1005, …). So there are 50 even addresses and 50 odd addresses.

Let us have 50 pigeonholes each containing a pair of consecutive addresses (even & odd). So we have pigeonhole1 having (1000 & 1001), pigeonhole2 having (1002, 1003) ,…, pigeonhole50 having (1098 & 1099).

Since there are 51 houses at two houses would go into the same pigeonhole. Then the even-odd combination would make up for at least two consecutive addresses.

(Observe that for the addresses not to be consecutive all addresses must be even OR all addresses must be odd).

26. Show that if there are 100,000,000 wage earners in the United States who earn less than 1,000,000 dollars then there are two who earned exactly the same amount, to the penny, last year.

100,000,000 wage earners earned between \$0.01 and \$999,999.99 dollars last year.
In other words 100,000,000 wage earners earned between 1 penny and 99,999,999 pennies last year.

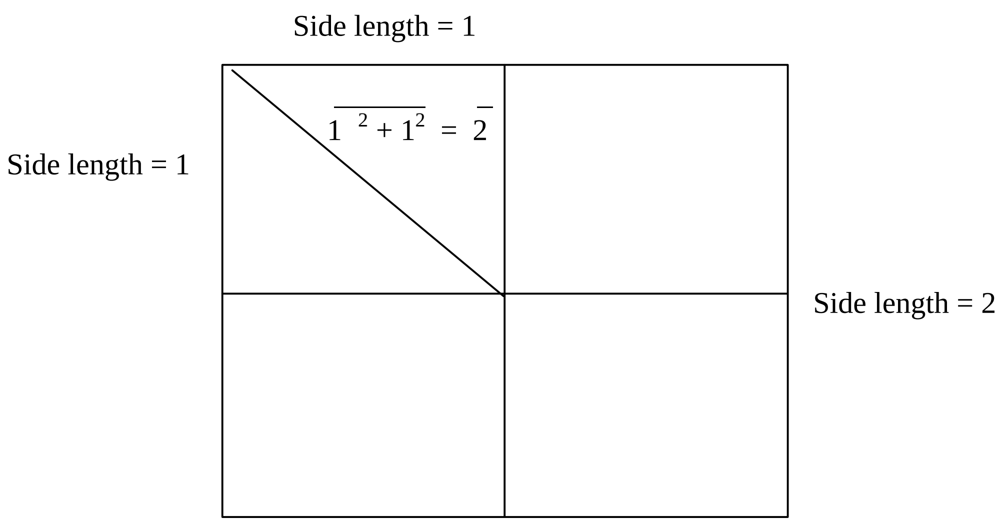
So the number of possible wages (to the penny) are = 99,999,999 – 1 + 1 = 99,999,999

The number of possible wages is just like the number of boxes.
There are 100,000,000 = 99,999,999 + 1 objects to be put into 99,999,999 boxes.

According to the pigeonhole principle at least two objects must go into the same box.
Hence there are two wage earners who earned exactly the same amount to the penny.

Page 303 (Supplementary Exercises) Exercise 18 (All Sections) Show that if five points are picked in the interior of a square of side length 2 then ate least two of these points are no farther than $\sqrt{2}$ apart.

Let's divide a square of side length 2 into four subdivided squares of side length 1 each.



Each subdivided square would have a diagonal of length $\sqrt{1^2 + 1^2} = \sqrt{2}$ units. (By the Pythagoras' Theorem). A diagonal is the maximum length between any two points inside a square.

If five points are chosen in a square of side length 2, at least two of them would lie inside a subdivided square. Hence the maximum distance between them would not be more than $\sqrt{2}$